

# Flow Chart

Action-Angle Variables



Tori (nested) ~~and~~ integrability



Perturbative Integrability?  
Resilience of Tori?



canonical P.T.

Small Divisor Problem  
Resonant Surfaces  
Rational



Secular P.T.

Island Formation  
by Resonant Pert.

KAM  
Theorem



conceptual step

Chirkov Crit. /  
Over/comp

study of systems → standard MAP

Chaos, Stochasticity



Calculation in Chaotic  
Regime ? ?

# Calculating in Chaotic Regime I

→ A First Introduction to Stat. Mech.

→ have discussed:

- problem of perturbative integrability
- small divisors and island formation
- KAM Theorem
- Chirikov criterion and onset chaos

now key question: How calculate in chaotic regime??

→ Statistical Mechanics  $\Rightarrow$   $\left\{ \begin{array}{l} \text{seek calculate} \\ \text{pdf of orbits} \end{array} \right.$

- chaos  $\leftrightarrow$  deterministic, random process

- approach via methodology of random processes:

- Mean Field / Quasilinear Theory
- Fokker-Planck Theory

- key approximation:

- fundamentally, orbit is stochastic
- but treat perturbation as random, even though it is not.

i.e. - off example,  $v: \Theta \rightarrow \text{phase}$

- 'know'  $\langle \tilde{B}_n^2 \rangle_{\mu, n}$  but not  $\tilde{B}_{nm}$

d.e.  $\left. \begin{matrix} \langle \tilde{B}_r / m_n \rangle \text{ fixed} \\ \tilde{B}_r / m_n = 0 \end{matrix} \right\} \Rightarrow \tilde{B}_r \text{ as Gaussian random variable (via phase)}$   
 $\Rightarrow$  RPA

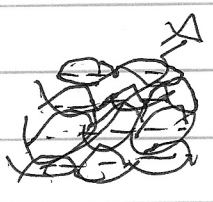
Random Phase Approx

- Seek Calculate:

- a) - diffusion coefficient  $D_f \rightarrow \langle (\Delta J)^2 \rangle / \Delta t$   
 $\rightarrow$  basic measure chaos
- b) - orbit divergence rate  $\rightarrow$  d.e. Lyapunov exponent  $h$ .

a) Field Line Diffusion - Hierarchy Transition <sup>Mech Field Theory</sup> } How far how fast do orbits hop in reduced space?

$f \equiv$  pdf of orbit in  $r, \theta, z$



d.e. recall:  $\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{dz}{B_z}$

strategy:  $\frac{\partial}{\partial z} \langle f \rangle$

obviously  $f$  constant along field, ~~so~~  
 so:

$B \cdot \nabla f = 0$

total  $f$  conserved along field  
 $\Leftrightarrow$  Liouville Eqn.

$$\Rightarrow \frac{\partial F}{\partial z} + \frac{B_\theta}{B_z} \frac{1}{r} \frac{\partial F}{\partial \theta} + \frac{B_r}{B_z} \frac{\partial F}{\partial r} = 0$$

$$\begin{cases} B_r = \tilde{B}_r \\ B_\theta = \langle B_\theta(r) \rangle + \tilde{B}_\theta \end{cases} \quad \nabla_\perp \cdot \underline{\tilde{B}}_\perp = 0$$

$\downarrow$  toroidal current       $\downarrow$  pert.

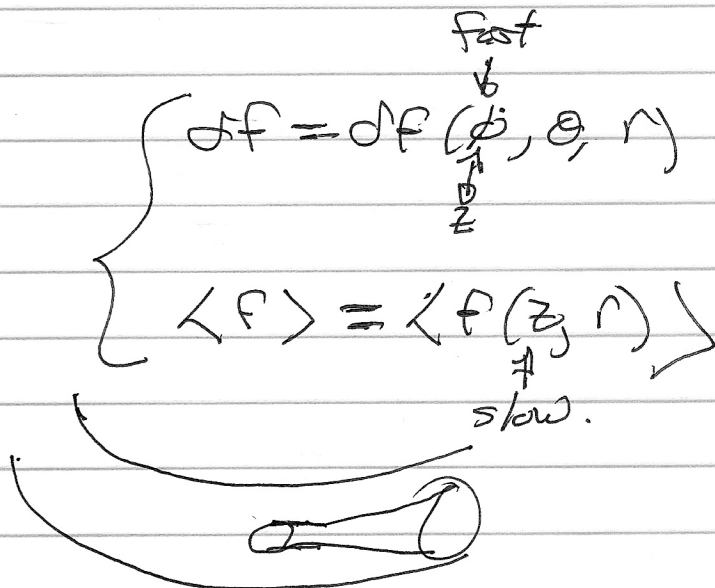
and

$$F = \delta F + \langle F \rangle$$

$\downarrow$   
Fluctuation induced by  $\tilde{B}$ .

$\downarrow$   
Mean dist.

$\downarrow$   
coarse graining!



$\Rightarrow$

$$\frac{\partial F}{\partial z} + \frac{\langle B_\theta \rangle}{B_z r} \frac{\partial F}{\partial \theta} + \frac{\partial}{\partial r} \left( \frac{\tilde{B}_r}{B_z} F \right)$$

$$+ \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\tilde{B}_\theta}{B_z} F \right) = 0$$

Now,  $\langle \rangle = \langle \rangle_{\theta, \phi}$

but retain slow dependence

- alternatively, ensemble.

→ energy:

$$\frac{\partial \langle f \rangle}{\partial z} + \left\langle \frac{\langle B_{\theta} \rangle}{B_z} \frac{\partial f}{\partial \theta} \right\rangle + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_z} \delta f \right\rangle = 0$$

→ Mean Field Equation:

$$\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_z} \delta f \right\rangle = 0$$

tracks slow variation in z due radial flux  $\langle f \rangle$

b. -  $\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial \Gamma}{\partial r} = 0$

div. flux → continuity equation

- hierarchy pb/m.

c.e.  $\delta f \leftrightarrow \tilde{B}_r$

$$\frac{\partial \delta f}{\partial z} + \frac{\langle B_{\theta} \rangle}{B_T} \frac{\partial \delta f}{\partial y} = - \frac{\tilde{B}_r}{B_T} \frac{\partial \langle f \rangle}{\partial r}$$

$$\tilde{B}_r = \sum_{m,n} \tilde{B}_{rn} e^{i(m\theta - n\phi)}$$

$$= \sum_{k_z, k_{\theta}} \tilde{B}_{\theta} e^{i(k_{\theta} y - k_z z)}$$

(slsb)

$$\text{so } \frac{\partial \langle F \rangle}{\partial z} \sim \begin{matrix} \mathcal{O}(\delta F^2) \\ \mathcal{O}(\hat{B}_r^2) \end{matrix}$$

$$\frac{\partial \langle \delta F^2 \rangle}{\partial z} \sim \mathcal{O}(\delta F^3)$$

$\left\{ \begin{array}{l} \infty \\ \text{of} \end{array} \right.$  hierarchy of moments  
 c.e. eqn. bilinear  
 how truncate?

- mean field theory / quasi-linear theory

- compute  $\left\langle \frac{\tilde{B}_r}{B_0} \delta F \right\rangle$  break average

by substituting linear response  $\delta F$  into flux - why??

- yields a mean  $\Gamma(r)$   $\rightarrow$  radial flux due stochastic wandering of lines.

- closes hierarchy, as:

$$\Gamma = -D_M \frac{\partial \langle F \rangle}{\partial r}$$

$\Rightarrow$  diffusion equation!

Fickian flux  
 $D_M \leftrightarrow$  counterpart of  $D \approx k^2/4$  for st. map  
 for  $\langle F \rangle$

$$\Gamma = \sum_{m,n} \frac{\tilde{B}_{r,m}}{\bar{B}_0^{-n}} dF_{m,n}$$

$$= \sum_n \frac{\tilde{B}_{r,n}}{\bar{B}_0} dF_n$$

Now,

$$i \left( k_y \frac{\langle B_0 \rangle}{B_T} - k_z \right) dF_n = - \frac{\tilde{B}_{r,n}}{\bar{B}_0} \frac{\partial \langle F \rangle}{\partial r}$$

⇒

$$dF_n = \frac{i}{\left[ k_z - k_y \frac{\langle B_0 \rangle}{B_T} + i\epsilon \right]} \left( - \frac{\tilde{B}_{r,n}}{\bar{B}_0} \frac{\partial \langle F \rangle}{\partial r} \right)$$

causality (damping at  $z \rightarrow -\infty$ )

⇒

$$\frac{i}{x + i\epsilon} = \epsilon \left( \frac{1}{x} - i\pi \delta(x) \right)$$

$$= \cancel{\frac{i}{x}} + \pi \delta(x)$$

⇒

$$\Gamma = - \sum_n \left| \frac{\tilde{B}_{r,n}}{\bar{B}_0} \right|^2 \pi \delta \left( k_z - k_y \frac{\langle B_0 \rangle}{B_T} \right) \frac{\partial \langle F \rangle}{\partial r}$$





$$\Gamma = -D_M \frac{\partial \langle F \rangle}{\partial r}$$

$$D_M = \sum_n \left| \frac{\tilde{B}_{r,n}}{B_0} \right|^2 \pi \delta \left( k_z - k_y \frac{\langle B_\theta \rangle}{B_T} \right)$$

$$= \sum_{m,n} \left| \frac{\tilde{B}_{m,n}}{B_0} \right|^2 R \pi \delta \left( n - \frac{m}{\epsilon(r)} \right)$$

$D_M \equiv$  Field line diffusion coefficient

and  $Q = L / \text{Mean Field Egn.}$

go to 6

$$\frac{\partial \langle F \rangle}{\partial z} = \frac{\partial}{\partial r} D_M \frac{\partial \langle F \rangle}{\partial r}$$

$\langle \Delta r^2 \rangle$   
 $\sim D_M z$

N.B.

$$D_M \sim L$$

- for standard map:

$$D \sim k^2 \quad T \sim l \approx \pm$$

parallel coherence length of scattering field

$$- \text{ here } D \sim \left( \frac{\tilde{B}_r}{B} \right)^2, \quad l_{\text{sc}} \sim \frac{1}{\Delta |k_{\parallel}|}$$

i.e.  $\frac{\partial \langle F \rangle}{\partial Z} = \frac{\partial}{\partial r} D_{rr} \frac{\partial \langle F \rangle}{\partial r} \Rightarrow \langle D_{rr} \rangle \sim D_{rr} Z$  7.

(very cryptic)

→ origin of irreversibility →  
chaos !, due island overlap

→ why diffusion ?

- trajectory can 'hop' either direction  
↑ radially → net flux set  
↓ by  $\partial \langle F \rangle / \partial r$ .

- diffusion measured:

mean square displ.

$$\langle (\Delta r)^2 \rangle / \Delta Z \quad \left( \rightarrow \frac{\langle (\Delta x)^2 \rangle}{\Delta t} \right)$$

random walk

$\Delta r \rightarrow$  step size, radius

↕  
chaos !

$\Delta Z \rightarrow Z$  increment (time step)

- why linear response valid ?

$h_{ac} < h_{\pm}$

(tba)

$h_{ac} < h_{\pm}$

i.e. scattering field pattern decorrelates rapidly → ensures:

$$\frac{l_{\text{coh}}}{\Delta r} \frac{\tilde{B}_r}{B_0} < 1 \Rightarrow \text{Kubo} \#$$

$$K \ll 1.$$

- What does Kubo # mean?

$\Delta r \rightarrow$  radial scale of scattering field

i.e.  $\tilde{B}_r(r) \leftrightarrow$  radial scale  
i.e.  $\tilde{B} = \tilde{B}(r/\Delta)$

$d r \rightarrow$  scattering excursion in  
radial

i.e.

$$\frac{d r}{d z} = \frac{\tilde{B}_r}{B_0} \Rightarrow d r \sim \int d z \frac{\tilde{B}_r}{B_0}$$

$l_{\text{coh}} \rightarrow$  parallel  
coherence  
length of  
scattering field

$$\sim l_{\text{coh}} \frac{\tilde{B}_r}{B_0}$$

then

$$\frac{d r}{\Delta} < 1 \rightarrow \text{weak scattering } (K \ll 1)$$

$\rightarrow$  many kicks in  $\Delta$   
radial scale of scatterer.

$\rightarrow$  linear response @ OH.

$$\frac{d r}{\Delta} > 1 \rightarrow \text{strong scattering } (K > 1)$$

$\rightarrow$  linear response fails.

$$\text{r. b.} : \rightarrow k \cdot B_0 = k_y \langle B_0 \rangle + k_z B_T$$

$$\rightarrow = k_y \frac{\langle B_0 \rangle}{B_T} - k_z$$

$\rightarrow \text{fac} \sim \frac{1}{\Delta k_{\text{rad}}}$   $\rightarrow$  inverse bandwidth of parallel wave-number spectrum.  
width of spectrum

so, back to why linear response?

$\rightarrow$  small kick, relative to radial scatterer correlation length!

$$\rightarrow \text{kick } dr \sim \text{fac} \frac{\tilde{B}_r}{B_0}, \quad \frac{dr}{dz} = \frac{\tilde{B}_r}{B_0}$$

$$dr \ll \Delta_L \rightarrow k_{\text{tr}} \ll 1 \text{ again!}$$

$$\rightarrow D \sim \langle dr^2 \rangle / \Delta Z$$

$$dr \sim \text{fac} \frac{\tilde{B}_r}{B_0}$$

$$\Delta Z \sim \text{fac}$$

$$\Rightarrow D \sim \text{fac} \langle (\tilde{B}_r/B_0)^2 \rangle \rightarrow D_M$$

check:

$$k_r^2 D_r \text{ vs } \Delta |k_{\text{eff}}|$$

⇒

$$\frac{D_r}{\Delta r^2} < |\Delta k_{\text{eff}}|$$

⇒

$$\frac{(\partial r / \partial z)^2 \rho_{\text{ec}}}{\Delta r^2} < |\Delta k_{\text{eff}}|$$

$$\left( \frac{\tilde{B}_r \rho_{\text{ec}}}{B_T^2 \Delta r} \right)^2 < 1 \Rightarrow \text{kubo!}$$

→ general structure of standard map paradigm

$$\frac{\partial F}{\partial t} + \underline{\omega(J)} \cdot \underline{\nabla}_{\theta} F - \frac{\partial \tilde{H}(J, \theta)}{\partial \theta} \cdot \frac{\partial F}{\partial J} = 0$$

$$\frac{\partial F}{\partial z} + \frac{\langle B_0 \rangle}{B_r} \frac{\partial F}{\partial \theta} + \frac{\tilde{B}}{B_r} \cdot \underline{\nabla} F = 0$$

$$\langle \omega(J) \rangle \leftrightarrow \frac{1}{RZ(r)}$$

$$J \leftrightarrow r$$

$$m\tilde{H} \leftrightarrow \frac{\tilde{B}}{B} \quad \tilde{H} \leftrightarrow \tilde{A}$$

and

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial J} \cdot \left[ \frac{\partial \tilde{H}(J, \theta)}{\partial \theta} \frac{\partial F}{\partial \theta} \right]$$

$$\text{and } -i(\omega - \underline{m} \cdot \underline{\omega}(J)) \frac{\partial F_m}{\partial t} = \left( \frac{\partial \tilde{H}}{\partial \theta} \right)_m \frac{\partial \langle F \rangle}{\partial J}$$

$\downarrow$   $\downarrow$   
 $\Omega$   $\downarrow$   
 Key resonance.

$$-c (\Omega - m \cdot \omega(\sigma)) \delta F_m$$

$$= c m \tilde{H}_m \frac{\delta \langle F \rangle}{\delta J}$$

need overlap of  $\frac{\Omega}{m} = \omega(\sigma)$  resonances

20

$$\delta F_m = \frac{c}{\Omega - m \omega(\sigma)} c m \tilde{H}_m \frac{\delta \langle F \rangle}{\delta J}$$

$$\approx \pi \delta(\Omega - m \omega(\sigma)) c m \tilde{H}_m \frac{\delta \langle F \rangle}{\delta J}$$

21

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial t} D_J \frac{\delta \langle F \rangle}{\delta J}$$

$$D_J = \sum_m m^2 |\tilde{H}_m|^2 \pi \delta(\Omega - m \omega(\sigma))$$

- field line correspondence with this general structure is obvious.

Now, another length scale emerges, from considering decorrelation of trajectory from linear one, due to scattering in  $J$

i.e.

$$\frac{d\theta}{dt} = \omega(J)$$

$$\frac{d}{dt} \delta\theta = \frac{\partial \omega}{\partial J} \delta J$$

excursion

$$\Rightarrow \delta\theta \sim \int \frac{\partial \omega}{\partial J} \delta J$$

$$\langle \delta\theta^2 \rangle \sim \left( \frac{\partial \omega}{\partial J} \right)^2 \langle \delta J^2 \rangle$$

$$\text{but } \langle \delta J^2 \rangle \sim D_J t$$

$$\langle \delta\theta^2 \rangle \sim \left( \frac{\partial \omega}{\partial J} \right)^2 D_J t^3$$

$\rightarrow$  enhanced decorrelation from scattering of action

i.e. not  $\sim t$



For F-O-M of scattering: use m of scatterer field.

$$m^2 \langle \sigma \sigma^2 \rangle \sim 1 \sim m^2 \left( \frac{\partial \omega}{\partial \tau} \right)^2 D_J t^3$$

$$\Rightarrow 1/l_c \sim \left[ m^2 \left( \frac{\partial \omega}{\partial \tau} \right)^2 D_J \right]^{1/3} \rightarrow \text{orbit decorrelation rate}$$

For lines:

$$1/l_{\perp} \sim m^2 \frac{1}{R^2} \frac{\Sigma^{1/2}}{\Sigma^2} D_M$$

$$\sim k\omega^2 \left( \frac{R\omega'}{E} \right)^2 \left( \frac{1}{R\Sigma} \right)^2 D_M$$

$$\sim k\omega^2 \left( \frac{\vec{s}}{R\Sigma} \right)^2 D_M$$

$\vec{s} = \frac{v\omega'}{2}$   
shear parameter.

$$\Rightarrow 1/l_{\perp} \sim \left[ k\omega^2 \left( \frac{\vec{s}}{R\Sigma} \right)^2 D_M \right]^{1/3}$$

length over which line is decorrelated by Br scattering by  $k\omega^{-1}$  from orbit.



Notes: Length scales:

$\Delta_{\perp} \rightarrow$  scatter scale, radius

$l_{ac} \sim \frac{1}{k_{\perp}} \rightarrow$  parallel coherence length of scatterer field.

$l_{\perp} \rightarrow$  decorrelation length for orbit.

then  $l_{ac} < l_{\perp} \Rightarrow$  scatter field must reset before orbit decorrelated  
 check:  $k_{\perp} < \pm$   
 $l_{ac}^3 < l_{\perp}^3 \rightarrow$  many kicks

$$l_{ac}^3 < \frac{1}{k_0^2 \beta^2} D_M$$

$$k_{\perp} \sim \frac{l_{ac} B_r}{\Delta_{\perp} B_0}$$

$$D_M < \frac{(R_g)^2}{\beta^2} \frac{1}{k_0^2} l_{ac}^3 < \pm$$

$$l_{ac}^2 \left(\frac{B_r}{B_0}\right)^2 < \frac{(R_g)^2}{\beta^2} \frac{1}{k_0^2} \frac{l_{ac}}{l_{ac}^3}$$

$$< \frac{L_S^2}{k_0^2} \frac{1}{l_{ac}^2}$$

out

$$\frac{L_s^2}{k_0^2 \Delta r^2} \sim \frac{1}{\Delta |k_{||}|^2}$$

$$\frac{S}{R_L} = 1/L_s$$

$$\frac{l_{ac}^2}{\Delta r^2} \left(\frac{B_r}{B_0}\right)^2 < \frac{L_s^2}{k_0^2 \Delta^2} \frac{1}{l_{ac}^2} = 1 \quad \checkmark$$

⇒  $l_{ac} < l_{\perp} \Leftrightarrow k_{\perp} > 1$

Now:

- have used mean field theory + hierarchy closure to ~~the~~ calculate statistical properties, in chaotic regime, i.e.
- derived and understood behavior of  $f \rightarrow$  distribution function  $\rightarrow$  of chaotic system.
- next:
  - Fourier - Plank approach
  - calculation of Lyapunov exponent.